THE VALIDITY OF ERLANG'S TRUNK-CONGESTION FORMULA.

By RAGNAR HOLM.

WE do not possess any exact general theory of the congestion in automatic exchanges, but formulæ of accuracy sufficient for practice have been deduced. Regarding it of some importance to clear up the points when the approximating steps are taken by the theory, I state the following:—

The main problem to be solved is:—

Problem A: A telephone traffic having on an average y calls per time unit taken equal to the mean call duration is to be dealt with by means of x trunks. We seek the probability $S_{(y, z, x)}$ that x trunks are engaged at a certain moment and the probability $V_{(y, z)}$ that a certain call will find all trunks engaged.

Mr. Erlang's main deductions are based on the following correct principle:—There must be such an equilibrium that, on an average, during a time unit the number of transitions from the state $(x-1)$, that is to say, the state in which $x-1$ trunks are engaged to the state (x) is equal to the number of transitions from the state (x) to the state $(x-1)$.

If we indicate by $B_{(x)}$ the average rate of calls beginning at a state (x) and by $E_{(x)}$ the density of the ends of calls during a state (x) it follows from the equilibrium principle:—

$$S_{(y, z, x-1)} . \quad B_{(x-1)} = S_{(y, z, x)} . \; E_{(x)} \qquad (1)$$

We notice that for all x values:

$$B_{(x)} = y \qquad (2)$$

It thus follows from (1):

$$y . S_{(y, z, x-1)} = S_{(y, z, x)} . \quad E_{(x)} \qquad (3)$$

We now consider two cases:—

I. Call durations following the exponential law of durations, which assumes that after a call has begun every instant is equally likely to contain its termination. Then the probability p of exceeding a given duration t is $p = \epsilon^{-1}/T$, when T = the average duration, or as $T = 1$,

$$p = \epsilon^{-t} \qquad (4)$$

II. Uniform call durations.

In the first case, when there are x calls in progress, the probability of a call termination during the time dt will be xdt, *i.e.*, $E_{(x)}$ is like x.

Thus (3) becomes

$$y . S_{(y, z, x-1)} = x . S_{(y, z, x)} \qquad (5)$$

This is a recurrent formula which, upon an application of the equation $\sum_{0}^{\infty} S_{(y,\infty,x)} = 1$, leads to the formula $S_{(y,\infty,x)} = \frac{e^{-y} \cdot y^x}{x}$ ....(6)

and applied to the equation $\sum_{0}^{z} S_{(y,z,x)} = 1$ gives

$$S_{(y,z,x)} = \frac{\frac{y^x}{\underline{|x}}}{1 + \frac{y}{\underline{|1}} + \frac{y^2}{\underline{|2}} + \ldots\ldots + \frac{y^z}{\underline{|z}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

for $x \leqq z$.

Hence can be deduced

$$V_{(y,z)} = S_{(y,z,z)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

(7) and (8) are the well-known formulæ by Erlang.* They are exactly valid for the case of the exponential law of call durations. Now this law may only be approximately realized by real calls, because there will exist a certain minimum of call duration, which in practice will always be exceeded. This minimum causes an advance of the call duration law in the direction against the extreme instance of uniform durations.

For simplicity we consider this case. I am going to show that now $E_{(x)}$ *can* differ from x, especially when $z \neq \infty$. We may limit the proof to such x values that $\frac{x}{z}$ is great.

Assume there are two groups of trunks A and B, A comprising a practically unlimited number of trunks and B possessing only z trunks. Both groups are tested simultaneously by exactly equal series of calls, which, if they are put through, result in call durations of the unit time length. The rate of arrival of calls may be y, and $\frac{y}{z}$ may be small. Owing to the short duration of every state $(z+n)$ in A and to their rareness it may be postulated, that the calls ending during such a state have, with a probability nearly equal to 1, commenced during a state x where $x \leqq z$. That means that these calls occurred in both cases A and B. Thus the density $E_{1(x)}$ of call terminations during a state $(z+n)$ in A will be the same as the simultaneous density of call terminations in B, at least their difference may be made negligible by choosing $\frac{z}{y}$ and $\frac{x}{z}$ great enough. As now the state (z) in B partly belongs to periods of time, in which, in A, a state $(z+n)$ is prevailing, it is obvious that the density $E_{2(z)}$ of call terminations during the state (z) in B is

* See A. K. Erlang, *P.O.E.E. Journal*, X., p. 189, 1917.

determined by z and n. If we assume, as Erlang does, that $E_{1(x)}$ is = for all x-values the density $E_{2(z)}$ must be $z+\zeta$, where $\zeta > 0$.

Thus here Erlang's postulation

$$E_{1(z)} = E_{2(z)} = z$$

must be wrong.

Some years ago (1916) I worked out a theory* of the congestion in automatic exchanges based upon the assumption of equal call durations. The theory had only an approximating character. I therefore made some experiments with an artificial telephone traffic in order to test the formulæ. The traffic in question was equivalent to a nearly normally dispersed traffic with uniform call durations. Some results are given in the following table† :—

	Measured		Calculated	
	I.	II.	according to Holm.	according to Erlang.
V(5, 5 ; 10)	—	.012	.0129	.029
V(7 ; 10)	.0425	.053	.0402	.079
V(10 ; 14)	.0235	.032	.0253	.053

My formula is in good agreement with the measurements, while Erlang's formula gives far too high values.

We may resume from the foregoing that we possess an exact theory (Erlang's) for the case I., and a good approximation (Holm's) for the case II. As the case I. has been shown‡ to be more close to the reality than II., I would now prefer Erlang's formula (8) of $V_{(y,z)}$ for practical purposes. But, it is true that the difference between Erlang's and my formula of $V_{(y,z)}$ is of no practical importance.

* Published some years later, see R. Holm: *Archiv für Elektrotechnik*, Vol. 8, p. 413, 1920, and R. Holm, *P.O.E.E. Journal*, Vol. 15, p. 22, 1922. Other telephone traffic probability questions are also treated here.

† See *P.O.E.E. Journal*, Vol. 15, p. 33, 1922.

‡ See "Bilag 1" in *Sandsynlighedsregningens Anvendelse i telefondrift*, by A. K. Erlang, Kopenhagen, 1922 (a lecture given in 1920).

TELEGRAPH AND TELEPHONE PLANT IN THE UNITED KINGDOM.

TELEPHONES AND WIRE MILEAGES THE PROPERTY OF AND MAINTAINED BY THE POST OFFICE IN EACH ENGINEERING DISTRICT AS AT 30TH SEPT., 1924.

Telephone Stations.	Overhead Wire Mileage.				Engineering District.	Underground Wire Mileage.			
	Telegraph.	Trunk.	Exchange.	Spare.		Telegraph.	Trunk.	Exchange.	Spare.
423,474	583	4,053	58,951	556	London	21,998	44,498	1,478,549	46,852
49,328	1,929	19,051	49,666	1,719	S. East	3,627	17,754	81,480	12,970
52,790	4,311	23,240	40,449	1,583	S. West	13,489	2,792	77,807	1,949
40,802	8,915	28,138	36,723	3,994	Eastern	13,451	18,388	42,095	28,103
71,813	8,792	40,493	48,430	3,237	N. Mid.	15,771	27,071	117,989	81,902
49,971	4,933	24,920	50,865	4,636	S. Mid.	10,366	11,861	97,606	84,939
47,476	5,098	27,262	41,578	1,964	S. Wales	5,304	12,800	70,402	24,460
72,002	8,534	23,301	39,938	4,819	N. Wales	11,764	25,008	129,310	36,368
118,834	2,998	16,742	46,612	3,355	S. Lancs.	12,076	51,691	290,296	43,269
71,146	6,415	28,642	43,280	2,642	N. East	7,582	20,253	141,043	23,586
46,715	3,897	22,634	37,490	2,273	N. West	7,336	24,026	84,174	21,393
37,349	2,580	14,981	22,911	2,106	North	2,588	8,050	58,357	7,902
16,967	4,824	5,661	10,977	214	Ireland N.	113	57	28,024	128
48,662	5,614	20,520	31,785	1,323	Scot. East	1,483	5,602	77,355	11,450
71,994	7,537	22,278	38,417	1,047	Scot. West	12,236	18,595	179,516	27,665
1,219,323	76,960	321,916	598,072	35,468	Totals.	139,184	288,446	2,954,003	452,936
1,195,777	77,359	318,581	591,218	35,544	Figures on 30th June, 1924.	136,551	274,904	2,862,899	444,790